1.4.1 Substitution - Reversing the Chain Rule

The Chain Rule of Differentation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something") is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

Equivalently

$$\int 2x \cos x^2 \, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x \cos x^2$ back to $\sin x^2$.

How Substitution Works

Example 19

Determine $\int 2x\sqrt{x^2+1}\,dx.$ $2x$ $\sqrt{x^2+1}$ is a product

Solution Notice that the integrand involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.

Chain Rule

$$\frac{d}{dx}\left[g(u)\right] = g'(u)\frac{du}{dx}$$

products

1. Introduce the notation u and set $u = x^2+1$.
2. Note $\dfrac{du}{dx} = 2x$; rewrite this as $du = 2x\,dx$.
3. Then

$$\int 2x\underbrace{\sqrt{x^2+1}}_{\sqrt{u}=u^{1/2}}\overbrace{dx}^{du} = \int \sqrt{x^2+1}(2x\,dx) = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

$$\int u^{1/2}\,du$$

4. So

$$\int 2x\sqrt{x^2+1}\,dx = \boxed{\frac{2}{3}(x^2+1)^{\frac{3}{2}}} + C.$$

Check by differentiating: $\frac{2}{3}\cdot\frac{3}{2}(x^2+1)^{1/2}(2x) = \sqrt{x^2+1}\;\;2x$

$$\int \underline{2x \sqrt{x^2+1}} \, dx$$

$$u = x^2 + 1$$

The problem is

$$\int 2x \sqrt{u} \, dx = \int \left(\boxed{\frac{du}{\cancel{dx}} \sqrt{u}} \right) \cancel{dx}$$

This is essentially the result of differentiating some expression involving u, with respect to x, using the chain rule.

$$\int \sqrt{u} \, du$$

Substitution and definite integrals

Example 20

Determine $\int_0^{\pi} \cos^3 x \sin x\, dx$ *(from 2015 Summer paper)*

Solution: Write $u = \cos x$. Then

$$\frac{du}{dx} = -\sin x, \quad du = -\sin x\, dx, \quad \sin x\, dx = -du.$$

Change variables: $\int_0^{\pi} \cos^3 x \sin x\, dx = -\int_{x=0}^{x=\pi} u^3\, du$. Limits of integration: When $x = 0$, $u = \cos x = \cos 0 = 1$. When $x = \pi$, $u = \cos x = \cos \pi = -1$. Our integral becomes:

$$-\int_{u=1}^{u=-1} u^3\, du = -\frac{u^4}{4}\Bigg|_{u=-1}^{u=1} = -\frac{1}{4} + \frac{(-1)^4}{4} = 0.$$

Example 21

Evaluate $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr.$

The derivative of $4+r^2$ resembles the numerator $5r$

Solution To find an antiderivative, let $\boxed{u = 4 + r^2}$.

Then $\boxed{\frac{du}{dr} = 2r}$, $du = \boxed{2r\,dr}$; $5r\,dr = \frac{5}{2}du$.

So

$$\int \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int \frac{1}{u^2}\,du = \frac{5}{2}\int u^{-2}\,du.$$

$\frac{5}{2}\, u^{-1}/_{-1}$

Thus

$$\int \frac{5r}{(4+r^2)^2}\,dr = -\frac{5}{2} \times \frac{1}{u} + C,$$

(Indefinite integral)

and we need to evaluate $-\frac{5}{2} \times \frac{1}{u}$ at $r = 0$ and at $r = 1$. We have two choices.

Two Choices

$-\frac{5}{2} \times \frac{1}{u}\Big|_{r=0}^{r=1}$ $u = 4 + r^2$

1. Write $u = 4 + r^2$ to obtain

$$\begin{aligned}
\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2} \times \frac{1}{4+r^2}\Big|_{r=0}^{r=1} \\
&= -\frac{5}{2} \times \frac{1}{4+1^2} - \left(-\frac{5}{2} \times \frac{1}{4+0^2}\right) \\
&= -\frac{5}{2} \times \frac{1}{5} + \frac{5}{2} \times \frac{1}{4} \\
&= \frac{1}{8}.
\end{aligned}$$

2. Alternatively, write the antiderivative as $-\frac{5}{2} \times \frac{1}{u}$ and replace the limits of integration with the corresponding values of u.
When $r = 0$ we have $u = 4 + 0^2 = 4$.
When $r = 1$ we have $u = 4 + 1^2 = 5$.
Thus

$$\begin{aligned}
\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2} \times \frac{1}{u}\bigg|_{u=4}^{u=5} \\
&= -\frac{5}{2} \times \frac{1}{5} - \left(-\frac{5}{2} \times \frac{1}{4}\right) \\
&= \frac{1}{8}.
\end{aligned}$$

From Summer Exam 2013

Example 22

Determine

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx.$$

Solution: Write

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx.$$

Now write $u = \sqrt{x}+1$. Then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} \Longrightarrow \frac{1}{\sqrt{x}}\,dx = 2du.$

Then

$$\begin{aligned} \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= \int_{x=1}^{x=4} \frac{2}{u}\,du = \int_{u=2}^{u=3} \frac{2}{u}\,du = 2\ln u\Big|_2^3 \\ &= 2(\ln 3 - \ln 2) = 2\ln\frac{3}{2}. \end{aligned}$$

Example 23

Determine $\int (1 - \cos t)^2 \sin t \, dt$

Question: *How do we know what expression to extract and refer to as u?*
Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.
There is no easy answer but with practice we can develop a sense of what might work. In this example the integrand involves the expression $1 - \cos t$ and also its derivative $\sin t$. This is what makes the substitution $u = 1 - \cos t$ effective for this problem.

NOTE: There are more examples of the substitution technique in the lecture notes.