Wednesday Feb 24

Recall from Week 3   The indefinite integral

$$\int f(x)\,dx = \boxed{F(x)+C},$$ "general antiderivative"

where $F(x)$ satisfies $F'(x) = f(x)$

For example $\boxed{\int 2x\,dx = x^2 + C}$

To calculate a definite integral $\boxed{\int_a^b f(x)\,dx}$

- find some $\boxed{F(x)}$ with $F'(x) = f(x)$
- Calculate $F(x)\Big|_a^b = F(b) - F(a)$ $x^2\Big|_2^5 = 5^2 - 2^2$

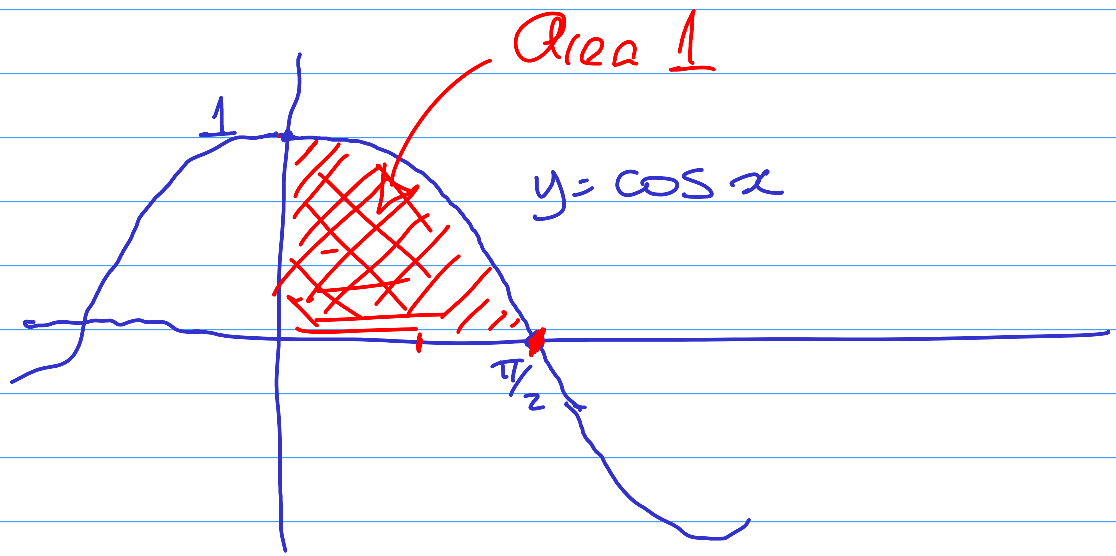
For example

$$\boxed{\int_2^5 2x\,dx = x^2\Big|_2^5 = 5^2 - 2^2 = 21}$$

$$\int_0^{\pi/2} \cos x \, dx = 1$$

$$\parallel$$

$$\sin x \Big|_0^{\pi/2} = \sin \pi/2 - \sin 0 = 1 - 0 = 1$$



Example 17

Determine $\int x^n \, dx$

" What do I need to differentiate to get x^n "

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\left[\frac{1}{n+1}x^{n+1}\right] = \frac{1}{n+1}(n+1)x^n = x^n$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

e.g. $\int x^5 \, dx = \frac{x^6}{6} + C$

$\int x^{1/2} \, dx = \frac{2}{3} x^{3/2} + C$

$\int x^{1/2} + x^2 \, dx = \frac{2}{3} x^{3/2} + \frac{x^3}{3} + C$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

Note:

This example includes $\int 1\,dx = \int x^0\,dx = x + C$

$$\int 5\,dx = 5x + C$$

$$\int \boxed{3x + 1}\,dx = 3\frac{x^2}{2} + x + C$$

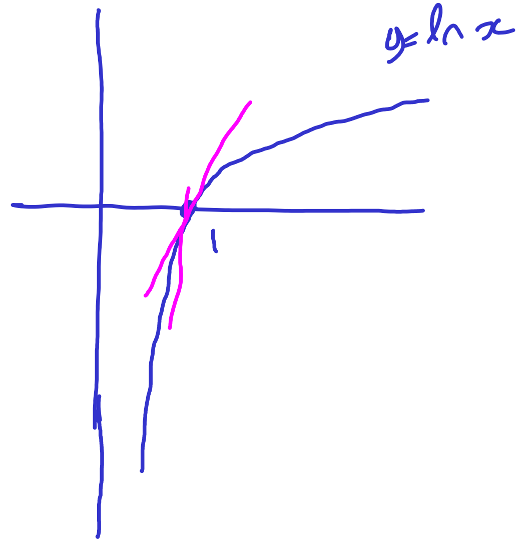
$$= \tfrac{3}{2}x^2 + x + C$$

The Integral of $\frac{1}{x}$ $= x^{-1}$

Suppose that $\boxed{x > 0}$ and $\boxed{y = \ln x}$. Recall this means (by definition) that $\boxed{e^y = x}$. Differentiating both sides of this equation (with respect to x) gives

$$e^y = x \implies \qquad e^y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \qquad \frac{d}{dx}[\ln x] = 1/x$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and



If $x<0$, then $\frac{d}{dx}[\ln(-x)] = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad \text{for } \boxed{x > 0}.$$

If $x < 0$, then

$$\boxed{\int \frac{1}{x}\,dx = \ln |x| + C.}$$

This latter formula applies for all $x \neq 0$.

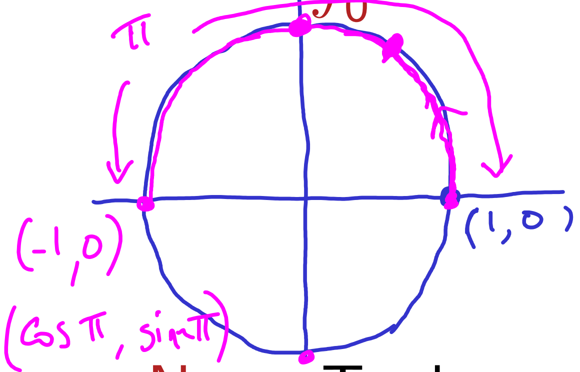
A definite integral

Example 18

Determine $\int_0^{\pi} \sin x + \cos x \, dx$.

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned}
\int_0^{\pi} \sin x + \cos x \, dx &= -\cos x + \sin x \Big|_0^{\pi} \\
&= (-\cos\pi + \sin\pi) - (-\cos 0 + \sin 0) \\
&= -(-1) + 0 - (-1 + 0) = 2.
\end{aligned}$$



Note: To determine $\cos\pi$, start at the point $(1, 0)$ and travel counter–clockwise along the the unit circle for a distance of π, arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos\pi$, and the y-coordinate is $\sin\pi$.

The Chain Rule of Differentation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something") is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x\cos x^2.$$

Equivalently

$$\int \boxed{2x\cos x^2}\, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x\cos x^2$ back to $\sin x^2$.

Example 19

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Solution Notice that the integrand involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.

1. Introduce the notation u and set $u = x^2+1$.
2. Note $\dfrac{du}{dx} = 2x$; rewrite this as $\boxed{du = 2x\,dx.}$
3. Then

$$\int 2x\sqrt{x^2+1}\,dx = \int \underline{\sqrt{x^2+1}}\,(\underline{2x\,dx}) = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

4. So

$$\int 2x\sqrt{x^2+1}\,dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$