Notes on the Fundamental Theorem

1. We won't formally prove the FToC, but to get a feeling for what it says, think about how $A(x)$ changes when x moves a little to the right. What if $f(x) = 0$? What if $f(x)$ is large/small/positive/negative?
2. The FToC is interesting because it connects differential calculus to the problem of calculating definite integrals, or areas under curves.
3. The FToC is useful because we know a lot about differential calculus. We can calculate the derivative of just about anything that can be written in terms of elementary functions. So we have a lot of theory about differentiation that is now relevant to calculating definite integrals as well.
4. The FToC can be traced back to work of *Isaac Barrow* and *Isaac Newton* in the mid 17th Century.

Calculating Definite Integrals

Finally we see how to use the FToC to calculate definite integrals.

Example 12

Calculate $\int_1^3 t^3 - t^2\,dt.$

Solution: Imagine that r is some point to the left of 1, and that the function A is defined for $x \geq r$ by

$$A(x) = \int_r^x t^3 - t^2\,dt.$$

Then

$$\int_1^3 t^3 - t^2\,dt = A(3) - A(1);$$

This is the area under the graph that is to the left of 3 but to the right of 1.

Example of a definite integral calculation (continued)

So: if we had a formula for $A(x)$, we could use it to evaluate this function at $x = 3$ and at $x = 1$.
What we know about the function $A(x)$, from the Fundamental Theorem of Calculus, is that its derivative is given by $A'(x) = x^3 - x^2$. What function A has derivative $x^3 - x^2$?

The derivative of x^4 is $4x^3$, so the derivative of $\frac{1}{4}x^4$ is x^3.

The derivative of x^3 is $3x^2$, so the derivative of $-\frac{1}{3}x^3$ is $-x^2$.

The derivative of $\frac{1}{4}x^4 - \frac{1}{3}x^3$ is $x^3 - x^2$.

Note : $\frac{1}{4}x^4 - \frac{1}{3}x^3$ is not the only expression whose derivative is $x^3 - x^2$.

For example $\frac{1}{4}x^4 - \frac{1}{3}x^3$ is another one, or anything of the form $\frac{1}{4}x^4 - \frac{1}{3}x^3 + C$, for any constant C. We only need one though.

Calculation of a definite integral

So: take $A(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3$. Then

$$\begin{aligned}
\int_1^3 t^3 - t^2 \, dt &= A(3) - A(1) \\
&= \left(\frac{1}{4}(3^4) - \frac{1}{3}(3^3)\right) - \left(\frac{1}{4}(1^4) - \frac{1}{3}(1^3)\right) \\
&= \frac{81-1}{4} - \frac{27-1}{3} \\
&= \frac{34}{3}.
\end{aligned}$$

Fundamental Theorem of Calculus, Part 2

This technique is described in general terms in the following version of the Fundamental Theorem of Calculus :

Theorem 13

(Fundamental Theorem of Calculus, Part 2)
Let f be a function. To calculate the definite integral

$$\int_a^b f(x)\,dx,$$

first find a function F whose derivative is f, i.e. for which $F'(x) = f(x)$. (This might be hard). Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Learning outcomes for Section 1.3

After studying this section, you should be able to

- Describe what is meant by an "area accumulation function".
- State the Fundamental Theorem of Calculus.
- Use the FToC to solve problems similar to Example 12 in these slides.
- Describe the general strategy for calculating a definite integral.
- Evaluate simple examples of definite integrals, like the one in Example 13 in these slides.

To calculate

$$\int_a^b f(x)dx$$

1. Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f.
2. Evaluate F at the limits of integration a and b; i.e. calcuate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
3. Calculate the number $F(b) - F(a)$. This is the definite integral $$\int_a^b f(x)\,dx.$$

Of the three steps above, the first one is the hard one.

Notation

Recall the following notation : if F is a function that satisfies $F'(x) = f(x)$, then

$$F(x)\big|_a^b \text{ or } F(x)\big|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

Definition 14

Let f be a function. Another function F is called an antiderivative of f if the derivative of F is f, i.e. if $F'(x) = f(x)$, for all (relevant) values of the variable x.

So for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about an antiderviative of a function or expression rather that the antiderivative.

The Indefinite Integral

Definition 15

Let f be a function. The indefinite integral of f, written

$$\int f(x)\,dx$$

is the "general antiderivative" of f. If $F(x)$ is a particular antiderivative of f, then we would write

$$\int f(x)\,dx = F(x) + C,$$

to indicate that the different antiderivatives of f look like $F(x) + C$, where C may be any constant. (In this context C is often referred to as a constant of integration).

Example 16

Determine $\int \cos 2x \, dx$.

Solution: The question is: what do we need to differentiate to get $\cos 2x$? Well, what do we need to differentiate to get something involving cos? The derivative of $\sin x$ is $\cos x$. A reasonable guess would say that the derivative of $\sin 2x$ might be "something like" $\cos 2x$.
By the chain rule, the derivative of $\sin 2x$ is in fact $2\cos 2x$.
So $\sin 2x$ is pretty close but it gives us twice what we want - we should compensate for this by taking $\frac{1}{2}\sin 2x$; its derivative is

$$\frac{1}{2}(2\cos 2x) = \cos 2x.$$

Conclusion: $\int \cos 2x \, dx = \frac{1}{2}\sin 2x + C$

Powers of x

Example 17

Determine $\int x^n \, dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x}\,dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

This latter formula applies for all $x \neq 0$.

A definite integral

Example 18

Determine $\int_0^{\pi} \sin x + \cos x \, dx$.

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned} \int_0^{\pi} \sin x + \cos x \, dx &= -\cos x + \sin x \Big|_0^{\pi} \\ &= (-\cos \pi + \sin \pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2. \end{aligned}$$

Note: To determine $\cos \pi$, start at the point $(1, 0)$ and travel counter–clockwise around the unit circle through an angle of π radians (180 degrees), arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos \pi$, and the y-coordinate is $\sin \pi$.