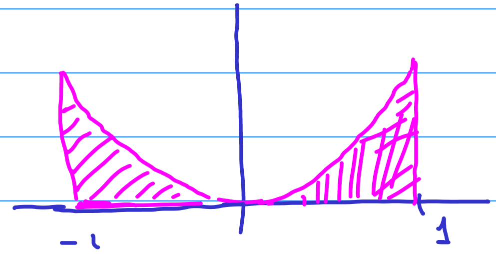
Zoom Poll

(1) $\int_{-1}^{1} x^2\,dx \neq 0$

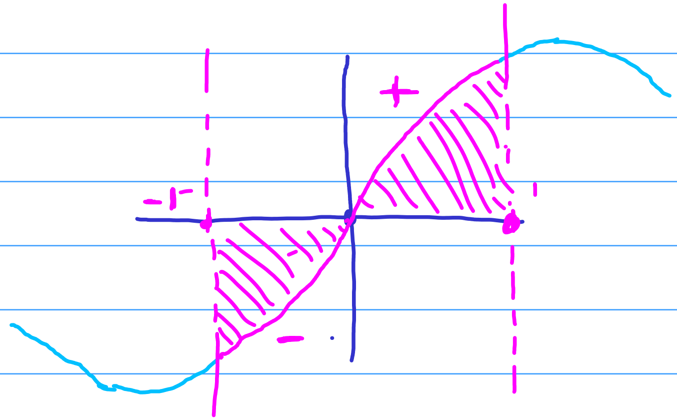


$\int_{-1}^{1} x^2\,dx > 0$

(since the region is above the x-axis)

(4) $\int_{-1}^{1} \sin x\,dx = 0$

because the "positive" and "negative" parts cancel each other



[sin is an odd function $\sin(-x) = -\sin x$]

Section 1.3: The Fundamental Theorem of Calculus

In this section, we discuss the Fundamental Theorem of Calculus which establishes a crucial link between differential calculus and the problem of calculating definite integrals, or areas under curves.

At the end of this section, you should be able to explain this connection and demonstrate with some examples how the techniques of differential calculus can be used to calculate definite integrals.

- Differential calculus is about *how functions are changing*.
- For example, temperature (in $^\circ C$) might be a function of time (in hours). Write temperature as $T(t)$ to indicate that the temperature T varies with time t. The derivative of the function $T(t)$, denoted $T'(t)$, tells us how the temperature is *changing over time*.
- If you know that at 10.00am yesterday the derivative of T was 0.5 ($^\circ$C/hr), then you know that the temperature was *increasing* by half a degree per hour at that time. However this does not tell you anything about what the temperature actually was at this time. If you know that by 10.00pm last night the derivative of the temperature was -2°C/hr you still don't know anything about what the temperature was at the time, but you know that it was cooling at a rate of 2 degrees per hour.

The “Area Accumulation” Function

Now we are going to define a new function related to definite integrals and consider its derivative.

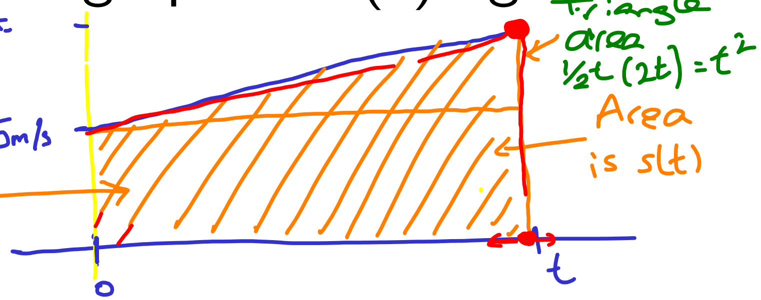
Example 9

At time $t = 0$ an object is travelling at 5 metres per second. After t seconds its speed in m/s is given by $v(t) = 5 + 2t$.

Let $s(t)$ be the distance travelled by the object after t seconds. From Section 1.1 we know that $s(t)$ is the area under the graph of $v(t)$ against t, between the vertical lines through 0 and t.

From the graph we can calculate $s(t) = 5t + t^2$.

rectangle of area $5t$



- $v(t) = \boxed{5 + 2t}$
- $s(t) = \boxed{5t + t^2}$ is the area under the graph $y = v(t)$, between 0 and t.
- Note $\boxed{s'(t) = 5 + 2t = v(t).}$

Derivative of $s(t)$ is $v(t)$

Important Note: The function $s(t)$ associates to t the area under the graph $y = v(t)$ from time 0 to time t. As t increases (i.e. as time passes), this area increases (it represents the distance travelled which is obviously increasing). Note that the derivative of $s(t)$ is exactly $v(t)$.

$$s(t) = 5t + t^2; \;\; s'(t) = 5 + 2t = v(t).$$

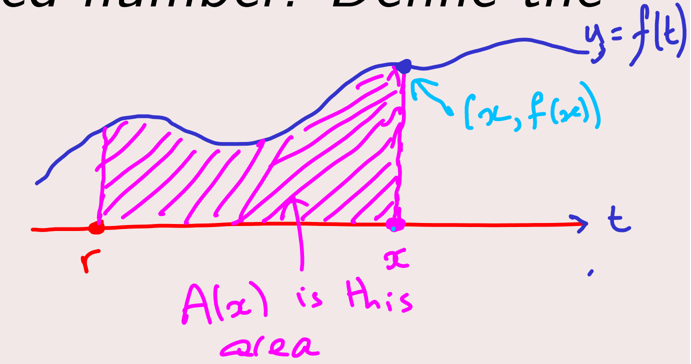
We shouldn't really be surprised by this given the physical context of the problem: $s(t)$ is the total distance travelled at time t, and $s'(t)$ at time t is $v(t)$, the speed at time t. So this is saying that the instantaneous rate of change of the distance travelled at a particular moment is the *speed* at which the object is travelling at that moment - which makes sense.

The Fundamental Theorem of Calculus

Theorem 10

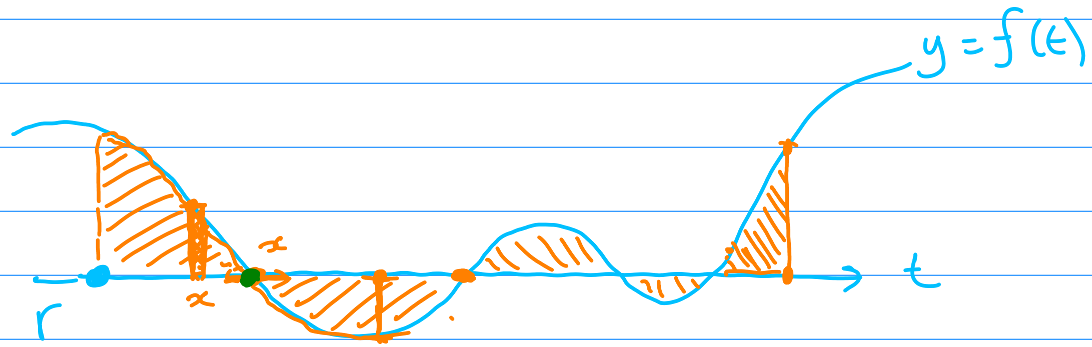
(The Fundamental Theorem of Calculus (FToC))
Let f be a (suitable) function, and let r be a fixed number. Define the *area accumulation function* *A by*

$$\boxed{A(x)} = \int_r^x f(t)\,dt.$$



This means: for a number x, $A(x)$ is the area enclosed by the graph of f and the x-axis, between the vertical lines through r and x.
The function A depends on the variable x, via the right limit in the definite integral. The Fundamental Theorem of Calculus tells us that the function f is exactly the derivative of this area accumulation function A. Thus

$$A'(x) = f(x).$$



A(x) is the definite integral indicated by the orange area

FToC: $A'(x) = f(x)$

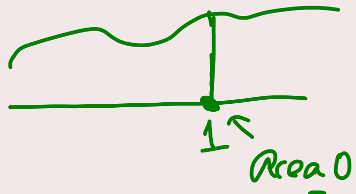
An Example (Summer 2016)

Example 11

Define a function A by

$$A(x) = \int_1^x (\cos t + \sin t)^3 \, dt,$$

$A(1) = \int_1^1 (\cos t + \sin t)^3 dt$



for $x \geq 1$.

$A'(1) = (\cos 1 + \sin 1)^3$

1. *What is* $A(1)$?
2. *Show that the function A is decreasing at* $x = \pi$.
   $\boxed{(2)\ A'(\pi) = (\cos \pi + \sin \pi)^3 = (-1+0)^3 < 0}$
3. *Show that* $A'(\frac{3\pi}{4}) = 0$. $A'(3\pi/4) = \left(\cos \frac{3\pi}{4} + \sin \frac{3\pi}{4}\right)^3 = \left(-1/\sqrt{2} + 1/\sqrt{2}\right)^3 = 0$

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1. (A) $(\cos 1 + \sin 1)^3$ (B) 0 ✓ (C) $(\cos t + \sin t)^3$ (D) 1.
2. We need to show that the derivative of A is negative at $x = \pi$.
3. This is a direct application of the FTOC.