

## Section 1.3: The Fundamental Theorem of Calculus

In this section, we discuss the **Fundamental Theorem of Calculus** which establishes a crucial link between differential calculus and the problem of calculating definite integrals, or areas under curves.

At the end of this section, you should be able to explain this connection and demonstrate with some examples how the techniques of differential calculus can be used to calculate definite integrals.

- Differential calculus is about *how functions are changing*.
- For example, temperature (in  $^{\circ}\text{C}$ ) might be a function of time (in hours). Write temperature as  $T(t)$  to indicate that the temperature  $T$  varies with time  $t$ . The **derivative** of the function  $T(t)$ , denoted  $T'(t)$ , tells us how the temperature is *changing over time*.
- If you know that at 10.00am yesterday the derivative of  $T$  was  $0.5$  ( $^{\circ}\text{C}/\text{hr}$ ), then you know that the temperature was *increasing* by half a degree per hour at that time. However this does not tell you anything about what the temperature actually was at this time. If you know that by 10.00pm last night the derivative of the temperature was  $-2^{\circ}\text{C}/\text{hr}$  you still don't know anything about what the temperature was at the time, but you know that it was cooling at a rate of 2 degrees per hour.

# The “Area Accumulation” Function

Now we are going to define a new function related to definite integrals and consider its derivative.

## Example 9

*At time  $t = 0$  an object is travelling at 5 metres per second. After  $t$  seconds its speed in m/s is given by  $v(t) = 5 + 2t$ .*

Let  $s(t)$  be the distance travelled by the object after  $t$  seconds. From Section 1.1 we know that  $s(t)$  is the area under the graph of  $v(t)$  against  $t$ , between the vertical lines through 0 and  $t$ .

From the graph we can calculate  $s(t) = 5t + t^2$ .

- $v(t) = 5 + 2t$
- $s(t) = 5t + t^2$  is the **area** under the graph  $y = v(t)$ , between 0 and  $t$ .
- Note  $s'(t) = 5 + 2t = v(t)$ .

# Derivative of $s(t)$ is $v(t)$

**Important Note:** The function  $s(t)$  associates to  $t$  the area under the graph  $y = v(t)$  from time 0 to time  $t$ . As  $t$  increases (i.e. as time passes), this area increases (it represents the distance travelled which is obviously increasing). Note that the derivative of  $s(t)$  is exactly  $v(t)$ .

$$s(t) = 5t + t^2; \quad s'(t) = 5 + 2t = v(t).$$

We shouldn't really be surprised by this given the physical context of the problem:  $s(t)$  is the total distance travelled at time  $t$ , and  $s'(t)$  at time  $t$  is  $v(t)$ , the speed at time  $t$ . So this is saying that the **instantaneous rate of change** of the distance travelled at a particular moment is the *speed* at which the object is travelling at that moment - which makes sense.

# The Fundamental Theorem of Calculus

## Theorem 10

*(The Fundamental Theorem of Calculus (FToC))*

*Let  $f$  be a (suitable) function, and let  $r$  be a fixed number. Define the **area accumulation function**  $A$  by*

$$A(x) = \int_r^x f(t) dt.$$

*This means: for a number  $x$ ,  $A(x)$  is the area enclosed by the graph of  $f$  and the  $x$ -axis, between the vertical lines through  $r$  and  $x$ .*

*The function  $A$  depends on the variable  $x$ , via the right limit in the definite integral. The Fundamental Theorem of Calculus tells us that the function  $f$  is exactly the derivative of this area accumulation function  $A$ .*

*Thus*

$$A'(x) = f(x).$$

# An Example (Summer 2016)

## Example 11

Define a function  $A$  by

$$A(x) = \int_1^x (\cos t + \sin t)^3 dt,$$

for  $x \geq 1$ .

- 1 What is  $A(1)$ ?
- 2 Show that the function  $A$  is decreasing at  $x = \pi$ .
- 3 Show that  $A'(\frac{3\pi}{4}) = 0$ .

- 1 (A)  $(\cos 1 + \sin 1)^3$  (B) 0 (C)  $(\cos t + \sin t)^3$  (D) 1.
- 2 We need to show that the derivative of  $A$  is negative at  $x = \pi$ .
- 3 This is a direct application of the FTC.

# Notes on the Fundamental Theorem

- 1 We won't formally prove the FToC, but to get a feeling for what it says, think about how  $A(x)$  changes when  $x$  moves a little to the right. What if  $f(x) = 0$ ? What if  $f(x)$  is large/small/positive/negative?
- 2 The FToC is **interesting** because it connects differential calculus to the problem of calculating definite integrals, or areas under curves.
- 3 The FToC is **useful** because we know a lot about differential calculus. We can calculate the derivative of just about anything that can be written in terms of elementary functions. So we have a lot of theory about differentiation that is now relevant to calculating definite integrals as well.
- 4 The FToC can be traced back to work of *Isaac Barrow* and *Isaac Newton* in the mid 17th Century.

# Calculating Definite Integrals

Finally we see how to use the FToC to calculate definite integrals.

## Example 12

Calculate  $\int_1^3 t^3 - t^2 dt$ .

**Solution:** Imagine that  $r$  is some point to the left of 1, and that the function  $A$  is defined for  $x \geq r$  by

$$A(x) = \int_r^x t^3 - t^2 dt.$$

Then

$$\int_1^3 t^3 - t^2 dt = A(3) - A(1);$$

This is the area under the graph that is to the left of 3 but to the right of 1.

# Example of a definite integral calculation (continued)

So: if we had a formula for  $A(x)$ , we could use it to evaluate this function at  $x = 3$  and at  $x = 1$ .

What we know about the function  $A(x)$ , from the Fundamental Theorem of Calculus, is that its derivative is given by  $A'(x) = x^3 - x^2$ . What function  $A$  has derivative  $x^3 - x^2$ ?

The derivative of  $x^4$  is  $4x^3$ , so the derivative of  $\frac{1}{4}x^4$  is  $x^3$ .

The derivative of  $x^3$  is  $3x^2$ , so the derivative of  $-\frac{1}{3}x^3$  is  $-x^2$ .

The derivative of  $\frac{1}{4}x^4 - \frac{1}{3}x^3$  is  $x^3 - x^2$ .

**Note** :  $\frac{1}{4}x^4 - \frac{1}{3}x^3$  is **not the only** expression whose derivative is  $x^3 - x^2$ .

For example  $\frac{1}{4}x^4 - \frac{1}{3}x^3$  is another one, or anything of the form

$\frac{1}{4}x^4 - \frac{1}{3}x^3 + C$ , for any constant  $C$ . We only need one though.

# Calculation of a definite integral

So: take  $A(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3$ . Then

$$\begin{aligned}\int_1^3 t^3 - t^2 dt &= A(3) - A(1) \\ &= \left( \frac{1}{4}(3^4) - \frac{1}{3}(3^3) \right) - \left( \frac{1}{4}(1^4) - \frac{1}{3}(1^3) \right) \\ &= \frac{81 - 1}{4} - \frac{27 - 1}{3} \\ &= \frac{34}{3}.\end{aligned}$$

# Fundamental Theorem of Calculus, Part 2

This technique is described in general terms in the following version of the Fundamental Theorem of Calculus :

## Theorem 13

*(Fundamental Theorem of Calculus, Part 2)*

*Let  $f$  be a function. To calculate the definite integral*

$$\int_a^b f(x) dx,$$

*first find a function  $F$  whose derivative is  $f$ , i.e. for which  $F'(x) = f(x)$ . (This might be hard). Then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

# Learning outcomes for Section 1.3

After studying this section, you should be able to

- Describe what is meant by an “area accumulation function”.
- State the Fundamental Theorem of Calculus.
- Use the FToC to solve problems similar to Example 12 in these slides.
- Describe the general strategy for calculating a definite integral.
- Evaluate simple examples of definite integrals, like the one in Example 13 in these slides.