Chapter 1: Integral Calculus

Section 1.1 Areas under curves - some examples

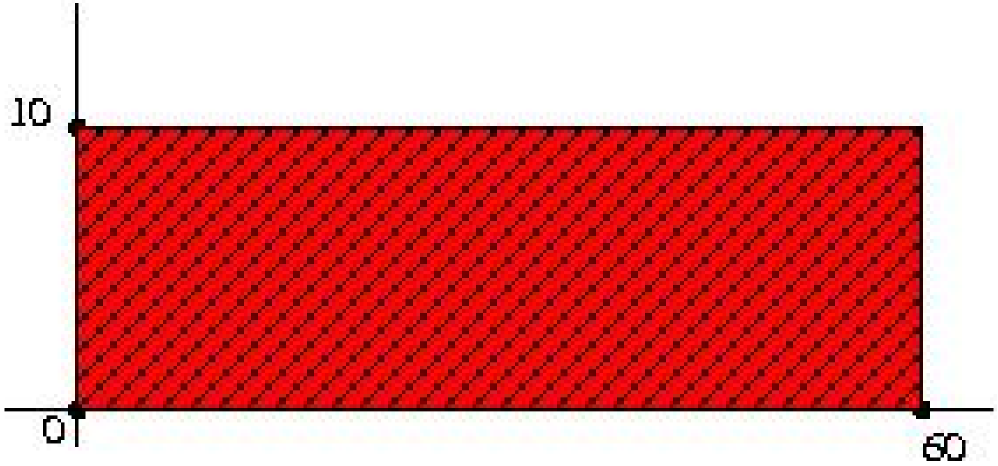
Problem 1

A car travels in a straight line for one minute, at a constant speed of 10 *m/s. How far has the car travelled in this minute?*

An easy example like this one can be a starting point for studying more complicated problems. What makes this example easy is that the car's speed is not changing so all we have to do is multiply the distance covered in one second by the number of seconds.

Graphical Interpretation

Suppose we draw a graph of the car's speed against time, where the x-axis is labelled in seconds and the y-axis in m/s. The graph is just the horizontal line $y = 10$ of course.



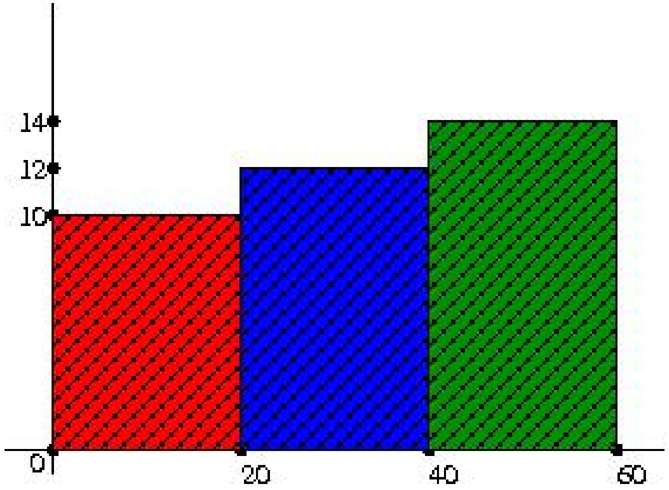
We label the time when we start observing the car's motion as $t = 0$ and the time when we stop as $t = 60$.
Note then that the total distance travelled – 600m – is the area enclosed under the graph, between the x-axis, the horizontal line $y = 10$, and the vertical lines $x = 0$ (or time $t = 0$) and $x = 60$ marking the beginning and end of the period of observation.

Another Problem

Problem 2

This time the car travels at 10 m/s for the first 20 seconds, at 12 m/s for the next 20 seconds, and at 14 m/s for the last 20 seconds. What is the total distance travelled?

This time, the picture is :



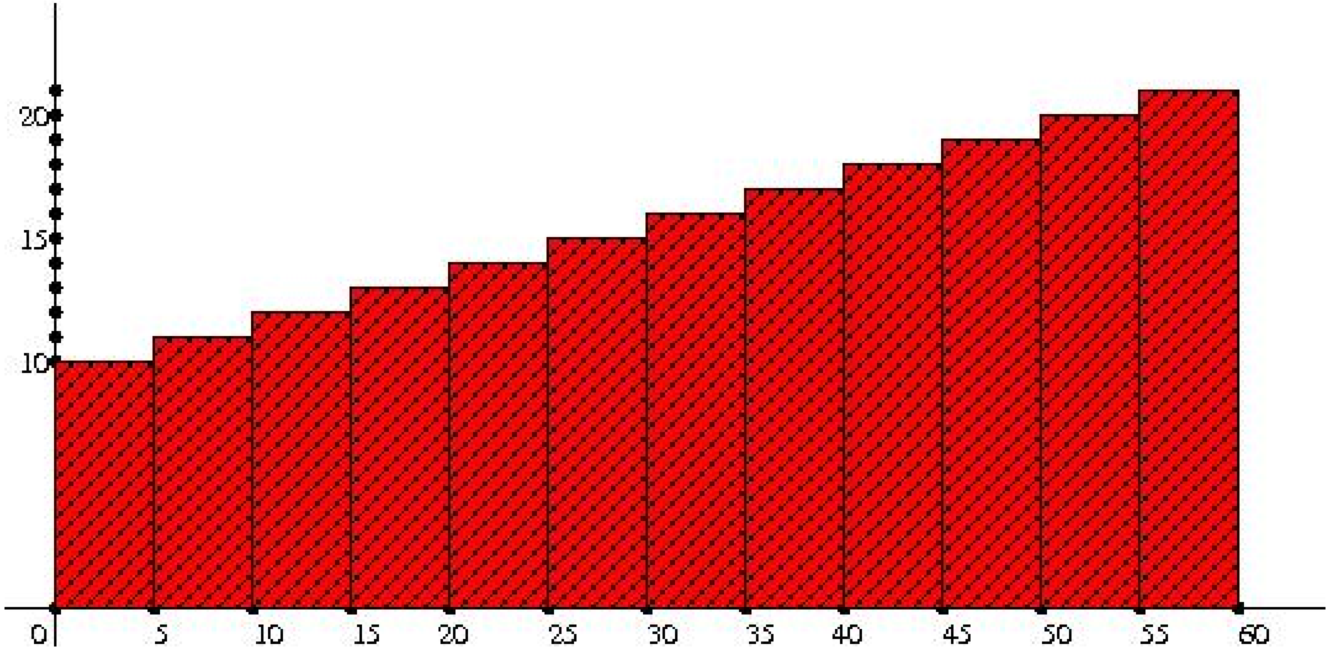
The total distance is again the area of the region enclosed between the lines $x = 0$, $x = 60$, the y-axis and the graph showing speed against time. The region whose area represents the distance travelled consists of three rectangles, all of width 20, and of heights 10, 12 and 14.

Another Problem

Problem 3

Same set up, but this time the car's speed is 10 m/s for the first 5 seconds, 11 m/s for the next 5, and so on, increasing by 1 m/s every five seconds so that the speed is 21 m/s for the last five seconds. Again the problem is to calculate the total distance travelled in metres.

The answer is left as an exercise, but this time the distance is the area indicated below.



A harder (but more realistic) problem

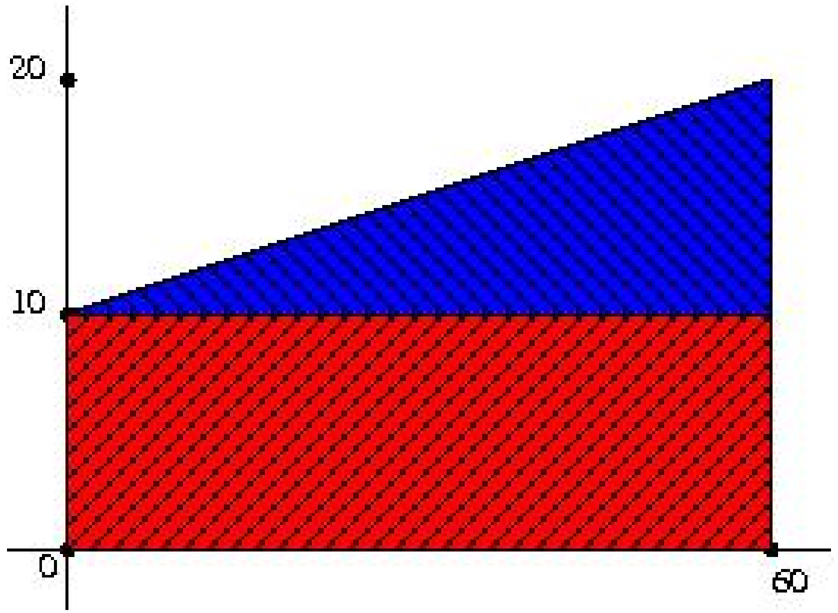
Problem 4

Again our car is travelling in one direction for one minute, but this time its speed increases at a constant rate from 10 m/s at the start of the minute, to 20 m/s at the end. What is the distance travelled?

Solution: This is a different problem, and more realistic. Because the speed is varying all the time this problem cannot be solved by just multiplying the speed by the time. However we can still consider the graph of the car's speed against time.

Distance travelled for continuously varying speed

Below is the graph of the speed against time.



If the total distance travelled is represented in this example, as in the case of constant speed, by the area under the speed graph between $t = 0$ and $t = 60$, then it is $60 \times 10 + \frac{1}{2}(60 \times 10) = 900$ metres.
Does this make sense?

Does this make sense?

Question 5

Just because the distance is given by the area under the graph when the speed is constant, how do we know the same applies in cases where the speed is varying continuously?

In the last problem, the speed increases steadily from 10 m/s to 20 m/s over the 60 seconds. We want to calculate the distance.

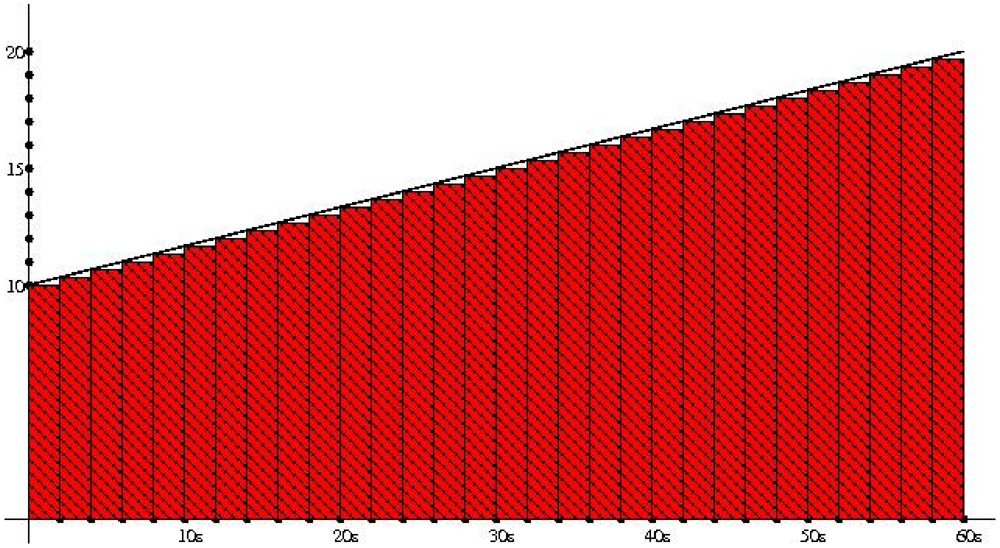
- Divide the one minute into 30 two-second intervals.
- At the start of the first two-second interval, the speed is 10 m/s. We make the simplifying assumption that the car travels at 10 m/s throughout these two seconds, covering 20 m in the first two seconds. This actually underestimates the true distance travelled in these two seconds, because in fact the speed is increasing from 20 m/s during this interval.

Continuously varying speed

- At the start of the 2nd two-second interval, the car has completed one-thirtieth of its acceleration from 10 m/s to 20 m/s, so its speed is
$$10 + \frac{10}{30} = 10\frac{1}{3}\mathrm{m/s}.$$
Make the simplifying assumption that the speed remains constant at $10\frac{1}{3}$ m/s throughout the second two-second interval.
- Proceeding like this we would estimate that the car travels $890m$ during the one-minute period.

A lower Riemann Sum

The distance that we estimate using the assumption that the speed remains constant for each of the 30 two-second intervals is indicated by the area in red in the diagram below, where the black line is the true speed graph. Note that the red area includes all the area under the speed graph, except for 30 small triangles of base length 2 and height $\frac{1}{3}$.



Refining the Estimate

- Suppose now that we refine the estimate by dividing our minute of time into 60 one-second intervals and assuming the the speed remains constant for each of these, instead of into 30 two-second intervals. This would give us a total of $895m$ as the estimate for distance travelled (check this). What is the corresponding picture?
- Note that this still underestimates the distance travelled in each second, for the same reason as before. But this estimate is closer to the true answer than the last one, because this estimate takes into account speed increases every second, instead of every two seconds.
- If we used the same strategy but dividing our minute into 120 half-second intervals, we would expect to get a better estimate again.

The True Answer

As the number of subintervals increases and their width decreases, the red rectangles in the picture come closer and closer to filling all the area under the speed graph. The true distance travelled is the limit of these improving estimates, as the length of the subintervals approaches zero. This is exactly the area under the speed graph, between $x = 0$ and $x = 60$.

We can assert more confidently now that the answer is 900 metres.

Note for independent study: With some careful attention you can check that if you split the one minute into n subintervals each of length $\frac{60}{n}$ and estimate the distance travelled as above, your answer will be $900 - \frac{300}{n}$. The limit of this expression as $n \to \infty$ is 900.

What if the area is harder to calculate?

Problem 6

Again our car is travelling in one direction for one minute, but this time its speed v *increases from 10 m/s to 20 m/s over the minute, according to the formula*

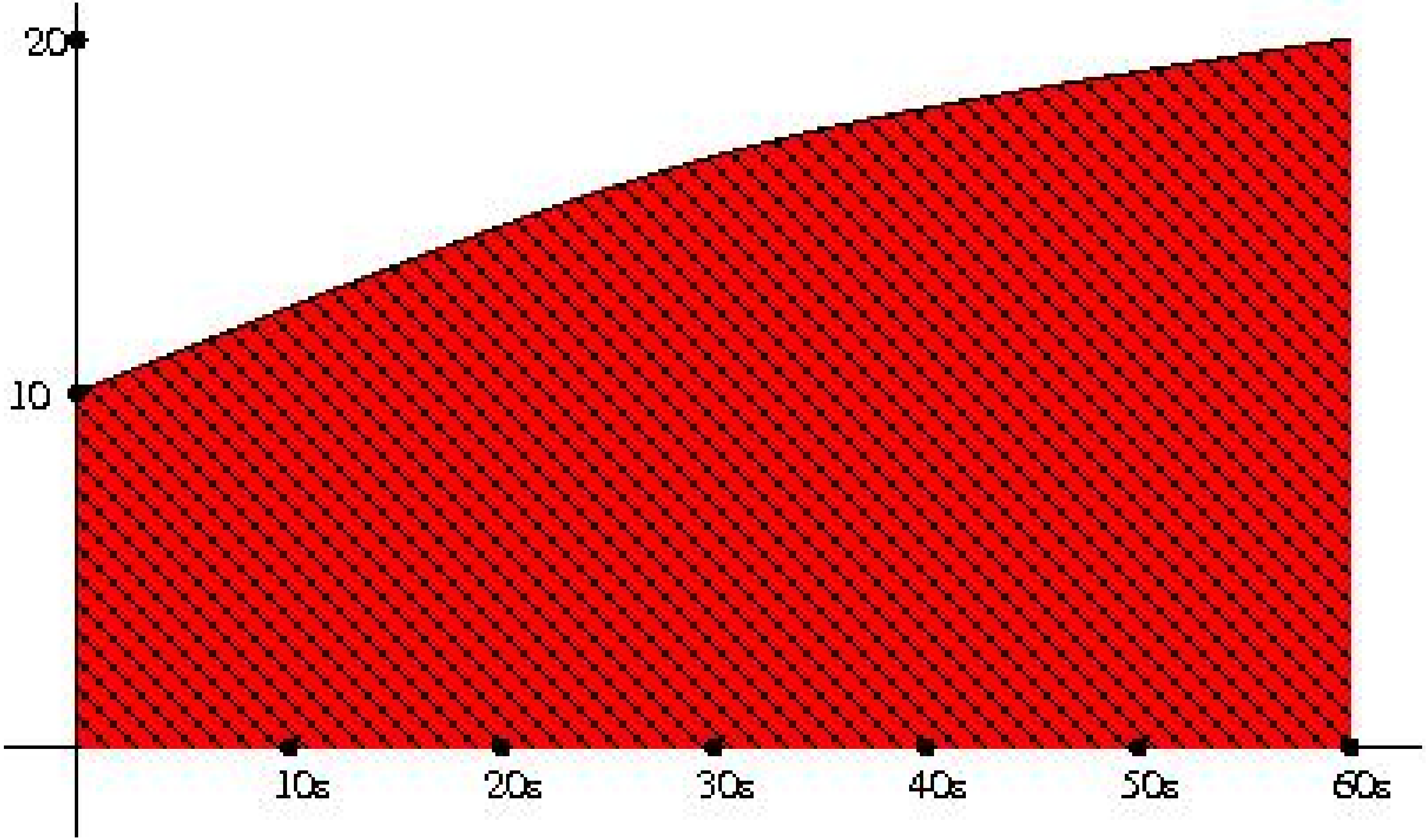
$$v(t) = 20 - \frac{1}{360}(60-t)^2,$$

where t *is measured in seconds, and* $t = 0$ *at the start of the minute. What is the distance travelled?*

Note: The formula means that after t seconds have passed, the speed of the car in m/s is $20 - \frac{1}{360}(60-t)^2$.

$v(t) = 20 - \frac{1}{360}(60 - t)^2$

Below is the graph of the speed (in m/s) against time (in s), with the area below it (between $t = 0$ and $t = 60$) coloured red.



A Harder Problem

The argument works in exactly the same way for this example, to persuade us that the distance travelled should be given by the area under the speed graph, between $t = 0$ and $t = 60$. This is the area that is coloured red in the picture.

Problem! The upper boundary of this area is a part of a parabola not a line segment. The region is not a combination of rectangles and triangles as in our earlier problems. We can't calculate its area using elementary techniques.

So: what we need is a theory or a method that will allow us to calculate the area bounded by a section of the graph of a function and the x-axis, over a specified interval.

Same Idea: Other Examples

Important Note: The problem of calculating the distance travelled by an object from knowledge of how its speed is changing is just one example of a scientific problem that can be solved by calculating the area of a region enclosed between a graph and the x-axis. Here are just a few more examples.

1. The fuel consumption of an aircraft is a function of its speed. The total amount of fuel consumed on a journey can be calculated as the area under the graph showing speed against time.
2. The energy stored by a solar panel is a function of the light intensity, which is itself a function of time. The total energy stored in one day can be modelled as the area under a graph of the light intensity against time for that day.

Same Idea : Other Examples

3. The volume of (for example) a square pyramid can be interpreted as the area of a graph of its horizontal cross-section area against height above the base.
4. In medicine, if a drug is administered intravenously, the quantity of the drug that is in the person's bloodstream can be calculated as the area under the graph of a function that depends both on the rate at which the drug is administered and on the rate at which it breaks down.
5. The concept of area under a graph is widely used in probability and statistics, where for example the probability that a randomly chosen person is aged between 20 and 30 years is the area under the graph of the appropriate probability density function, over the relevant interval.

Learning Outcomes from Section 1.1

After studying this section, you should be able to

- Explain, with reference to examples, why it is of interest to be able to calculate the area under the graph of a function over some specified interval.
- Explain how such an area can be approximated using rectangles, and how closer approximations can be obtained by taking narrower rectangles and using more of them.
- Solve simple problems similar to Problems 3 and 4 in this section.